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ON THE STRESS ANALYSIS OF OVERLAPPING BONDED ELASTIC SHEETS†

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Abstract—This investigation deals with the load-transfer between two overlapping, continuously bonded, elastic sheets of different thickness and distinct material properties, under a given in-plane loading. It is shown first that the foregoing stress-analysis problem is—within the theory of generalized plane stress—reducible to an elastic-inclusion problem. This general reduction scheme is subsequently applied to the specific problem concerning the transfer of loading between two overlapping semi-infinite sheets, attached to each other along a common strip adjacent to their edges, if one of the sheets is subjected to an internal concentrated force at right angles to its boundary. The solution obtained for this problem is studied in detail with particular attention to the quantitative appraisal of the bond forces acting throughout the interior of the region of adhesion and those communicated by the edges of the two sheets under consideration.

INTRODUCTION

THERE is little need to dwell on the importance in structural design of problems occasioned by the transfer of load from one elastic member to another, such as those arising in connection with lap-joined elastic plates or plate-stringer assemblies. The present investigation aims at a particular class of plane load-transfer problems: we are concerned here with two homogeneous and isotropic, overlapping elastic sheets of not necessarily the same thickness and possibly distinct material properties, which are continuously joined in perfect bond throughout their overlapping parts and are subjected to in-plane loads along the unattached portions of their periphery. Barring significant three-dimensional or bending effects, we limit our objective to an analysis of the stresses and deformations in the composite assembly within the conventional theory of generalized plane stress.

The class of problems described above was considered previously by Goodier and Hsu [1] in preparation for their attempt to cope with the diffusion of load from a transverse tension-bar into a semi-infinite elastic sheet.‡ As shown in [1], if both sheets have the same Poisson ratio, the bond tractions vanish identically throughout the interior of the region of adhesion, the entire load being transmitted by bond forces confined to the periphery of the overlapping sheet domain.

It is the main purpose of this paper to extend the general part of the analysis contained in [1] by admitting the possibility that the two sheets possess distinct Poisson ratios and to examine the quantitative influence of differences in this material parameter upon the mechanism of load-transfer. Specifically we aim at the role played by the *interior* bond tractions (which are now no longer absent) as compared to the bond forces communicated at the *edge* of the region of adhesion.

† The results communicated in this paper were obtained in the course of an investigation conducted under Contract Nonr-220(58) with the Office of Naval Research in Washington, D.C.

‡ See [2] for references to other investigations of the plate-stringer problem dealt with in [1], as well as to studies of related problems.

With this objective in mind we show first in Section 1 that every plane load-transfer problem of the type under present consideration is reducible, within the framework of the theory of generalized plane stress, to a plane inclusion problem, involving ordinarily an assembly of three distinct materials: the inclusion here alluded to occupies the overlapping sheet domain; its thickness and elastic constants are related in an elementary manner to those of the two sheets in the original problem. This analogy between the adhesion problem of overlapping sheets and an inclusion problem is useful for at least two reasons: it affords a convenient systematic method of attack upon the plane load-transfer problem that constitutes our objective; it enables one to reinterpret available results for inclusion problems as solutions of associated adhesion problems. In addition, the analogy is apt to be of some interest to experimenters.

In Section 2 we apply the reduction scheme established in the previous section (modified to accommodate interior loads), for the purpose of illustration, to a particular load-transfer problem involving two semi-infinite sheets (elastic half-planes) that are fastened to each other along an overlapping strip parallel and adjacent to their bounding edges. We assume the loading external to the composite body to consist of a single concentrated force acting at an interior point of one of the unattached sheet domains, at right angles to the bonded strip. The exact solution of this problem, which is deduced in integral form with the aid of the exponential Fourier transform, is discussed extensively in Section 3. Here we present illustrative numerical results displaying the quantitative influence of the relative material properties upon the interior and edge bond-forces. The example treated was chosen for its comparative simplicity and because of its relevance to the stress analysis of lap-joined assemblies.

1. A CLASS OF PLANE LOAD-TRANSFER PROBLEMS. INCLUSION ANALOGY

We now formulate, within the conventional theory of generalized plane stress, the class of plane load-transfer problems to be considered. For this purpose let S' and S'' be two, homogeneous and isotropic, linearly elastic sheets. Let D' and D'', with the boundaries C' and C'', be the open plane regions occupied by the interior of the middle section of S' and S'', respectively (Fig. 1). We assume for the present† that both D' and D'' are *bounded*

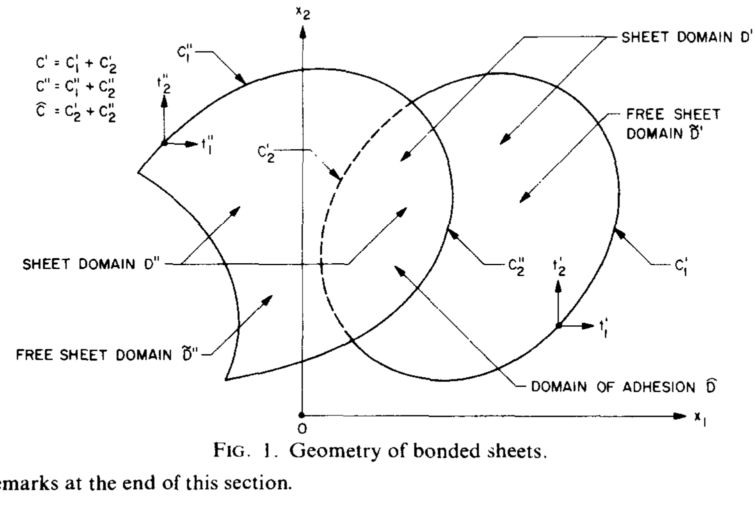

FIG. 1. Geometry of bonded sheets.

† See the remarks at the end of this section.

domains, while C' and C'' are simple closed curves. Suppose D' and D'' partly overlapping and call $\hat{D}$ the intersection of D' and D''. For the sake of simplicity we stipulate *perfect bond* between S' and S'' *throughout* $\hat{D}$ and therefore refer to $\hat{D}$ as the domain of adhesion.† Let $\tilde{D}'$ and $\tilde{D}''$ be the respective (open) *unattached sheet regions*; use C_1', C_1'' to designate the free—and C_2', C_2'' the bonded portions of C' and C'' (Fig. 1). Finally, call h', μ', ν' and h'', μ'', ν'', in this order, the uniform thickness, the shear modulus, and Poisson's ratio of the sheets S' and S'', respectively.

The composite assembly of S' and S'' is to be subjected to external loads that consist exclusively of surface tractions which are parallel to the sheet faces and are confined to the unattached boundary-components C_1' and C_1''. We refer the state of deformation and stress induced by the given loads to a plane rectangular cartesian coordinate system (x_1, x_2), chosen parallel to the sheet faces, and use indicial notation with the understanding that Greek subscripts range over the integers (1, 2). Thus u_α', $\tau_{\alpha\beta}'$ and u_α'', $\tau_{\alpha\beta}''$ represent the usual thickness averages of the cartesian components of displacement and stress appropriate to S' and S''. Further, to distinguish between field quantities defined on the free and the bonded sheet domains, we introduce the notation

$$\left.\begin{aligned} (u_\alpha', \tau_{\alpha\beta}') &= \begin{cases} (\tilde{u}_\alpha', \tilde{\tau}_{\alpha\beta}') \text{ on } \tilde{D}' \\ (\hat{u}_\alpha', \hat{\tau}_{\alpha\beta}') \text{ on } \hat{D}, \end{cases} \\ (u_\alpha'', \tau_{\alpha\beta}'') &= \begin{cases} (\tilde{u}_\alpha'', \tilde{\tau}_{\alpha\beta}'') \text{ on } \tilde{D}'' \\ (\hat{u}_\alpha'', \hat{\tau}_{\alpha\beta}'') \text{ on } \hat{D}. \end{cases} \end{aligned}\right\} \tag{1.1}$$

The stress equations of equilibrium and the displacement–stress relations in the theory of generalized plane stress then furnish the *field equations*

$$\left.\begin{aligned} \tilde{\tau}_{\alpha\beta,\beta}' &= 0, \quad \tilde{\tau}_{\alpha\beta}' = 2\mu'\left[\tilde{u}_{(\alpha,\beta)}' + \frac{\nu'}{1-\nu'}\delta_{\alpha\beta}\tilde{u}_{\gamma,\gamma}'\right] \text{ on } \tilde{D}', \\ \tilde{\tau}_{\alpha\beta,\beta}'' &= 0, \quad \tilde{\tau}_{\alpha\beta}'' = 2\mu''\left[\tilde{u}_{(\alpha,\beta)}'' + \frac{\nu''}{1-\nu''}\delta_{\alpha\beta}\tilde{u}_{\gamma,\gamma}''\right] \text{ on } \tilde{D}'', \end{aligned}\right\} \tag{1.2}$$‡

$$\left.\begin{aligned} \hat{\tau}_{\alpha\beta,\beta}' - \frac{f_\alpha}{h'} &= 0, \quad \hat{\tau}_{\alpha\beta}' = 2\mu'\left[\hat{u}_{(\alpha,\beta)}' + \frac{\nu'}{1-\nu'}\delta_{\alpha\beta}\hat{u}_{\gamma,\gamma}'\right] \text{ on } \hat{D}, \\ \hat{\tau}_{\alpha\beta,\beta}'' + \frac{f_\alpha}{h''} &= 0, \quad \hat{\tau}_{\alpha\beta}'' = 2\mu''\left[\hat{u}_{(\alpha,\beta)}'' + \frac{\nu''}{1-\nu''}\delta_{\alpha\beta}\hat{u}_{\gamma,\gamma}''\right] \text{ on } \hat{D}. \end{aligned}\right\} \tag{1.3}$$

Here $\delta_{\alpha\beta}$ is the Kronecker-delta, while f_α denotes the surface density components of the interior bond forces exerted by S' on S'' over the region of adhesion $\hat{D}$. These bond forces enter (1.3) in the role of reactive body forces and are among the unknown field quantities.

In view of the perfect adhesion between S' and S'' throughout $\hat{D}$, one has the *bond conditions*

$$\hat{u}_\alpha' = \hat{u}_\alpha'' \text{ on } \hat{D}, \tag{1.4}$$

† The generalization of the subsequent analysis to the case in which the two sheets are attached to each other merely over subregions of $\hat{D}$ is elementary.

‡ We employ the usual summation and differentiation conventions. Subscripts in parentheses refer to the symmetric part of the corresponding tensor components.

which are to be accompanied by the *boundary conditions*

$$\tilde{\tau}'_{\alpha\beta} n_\beta = t'_\alpha \text{ on } C'_1, \qquad \tilde{\tau}''_{\alpha\beta} n_\beta = t''_\alpha \text{ on } C''_1, \tag{1.5}$$

where n_β are the components of the unit outward normal of C' and C'', while t'_α and t''_α are the prescribed traction components. Finally, to (1.4) and (1.5) one needs to adjoin the *continuity conditions*

$$\left.\begin{aligned} \hat{u}'_\alpha &= \tilde{u}'_\alpha, \qquad (h'\hat{\tau}'_{\alpha\beta} + h''\hat{\tau}''_{\alpha\beta}) n_\beta = h'\tilde{\tau}'_{\alpha\beta} n_\beta \text{ on } C''_2, \\ \hat{u}''_\alpha &= \tilde{u}''_\alpha, \qquad (h'\hat{\tau}'_{\alpha\beta} + h''\hat{\tau}''_{\alpha\beta}) n_\beta = h''\tilde{\tau}''_{\alpha\beta} n_\beta \text{ on } C'_2, \end{aligned}\right\} \tag{1.6}$$

in which n_β retains its previous meaning.

The load-transfer problem under consideration thus consists in solving the twenty-two equations (1.2), (1.3), (1.4) for the twenty-two unknowns $\tilde{u}'_\alpha$, $\tilde{\tau}'_{\alpha\beta}$, $\hat{u}'_\alpha$, $\hat{\tau}'_{\alpha\beta}$ and $\tilde{u}''_\alpha$, $\tilde{\tau}''_{\alpha\beta}$, $\hat{u}''_\alpha$, $\hat{\tau}''_{\alpha\beta}$, f_α, subject to the boundary conditions (1.5) and the continuity conditions (1.6). Since D' and D'' are both finite, a necessary condition for the existence of a solution to this problem is that the entire given external loading be self-equilibrated, i.e.

$$\left.\begin{aligned} h' \int_{C'_1} t'_\alpha \, ds + h'' \int_{C''_1} t''_\alpha \, ds &= 0, \\ h' \int_{C'_1} \varepsilon_{\alpha\beta} x_\alpha t'_\beta \, ds + h'' \int_{C''_1} \varepsilon_{\alpha\beta} x_\alpha t''_\beta \, ds &= 0, \end{aligned}\right\} \tag{1.7}$$

where $\varepsilon_{\alpha\beta}$ are the components of the two-dimensional alternator.†

Let

$$\hat{\tau}'_{\alpha\beta} n_\beta = \hat{t}'_\alpha \text{ on } C'_2, \qquad \hat{\tau}''_{\alpha\beta} n_\beta = \hat{t}''_\alpha \text{ on } C''_2, \tag{1.8}$$

so that $\hat{t}'_\alpha$ and $\hat{t}''_\alpha$ denote the surface tractions acting on S' and S'' along the *bonded* portions of the corresponding sheet boundaries. It is essential to note that these tractions are reactive in nature and cannot be assigned in advance‡: $\hat{t}'_\alpha$ and $\hat{t}''_\alpha$ represent the *edge bond-tractions*—in contrast to the interior bond tractions—exerted on S' and S'', respectively.

Our next objective is the reduction of the adhesion problem formulated above to a standard second boundary-value problem (loads prescribed) for a composite body. To this end define fields u_α and $\tau_{\alpha\beta}$ through

$$u_\alpha = \hat{u}'_\alpha = \hat{u}''_\alpha, \; \tau_{\alpha\beta} = \frac{1}{h}(h'\hat{\tau}'_{\alpha\beta} + h''\hat{\tau}''_{\alpha\beta}) \qquad \text{on } \hat{D}, \tag{1.9§}$$

with

$$h = h' + h'', \tag{1.10}$$

and introduce the auxialiary elastic constants

$$\mu = \frac{\mu' h' + \mu'' h''}{h' + h''}, \qquad \nu = \frac{\nu'(1-\nu'') + \nu''(1-\nu')\rho}{1 - \nu'' + (1-\nu')\rho}, \tag{1.11}$$

† Thus $\varepsilon_{12} = -\varepsilon_{21} = 1$, $\varepsilon_{11} = \varepsilon_{22} = 0$.

‡ It will become apparent later that such an assignment would result in an over-determinate problem.

§ Recall the bond conditions (1.4).

where ρ is the "stiffness ratio" given by

$$\rho = \mu''h''/\mu'h'. \tag{1.12}$$

Then (1.3) at once imply

$$\tau_{\alpha\beta,\beta} = 0,\ \tau_{\alpha\beta} = 2\mu\left[u_{(\alpha,\beta)} + \frac{\nu}{1-\nu}\delta_{\alpha\beta}u_{\gamma,\gamma}\right] \quad \text{on } \hat{D}, \tag{1.13}$$

while (1.6) become

$$\left.\begin{aligned} u_\alpha &= \tilde{u}'_\alpha,\ h\tau_{\alpha\beta}n_\beta = h'\tilde{\tau}'_{\alpha\beta}n_\beta \quad \text{on } C''_2,\\ u_\alpha &= \tilde{u}''_\alpha,\ h\tau_{\alpha\beta}n_\beta = h''\tilde{\tau}''_{\alpha\beta}n_\beta \quad \text{on } C'_2. \end{aligned}\right\} \tag{1.14}$$

Consider now the fifteen field equations (1.2), (1.13)—in the fifteen unknowns $\tilde{u}'_\alpha$, $\tilde{\tau}'_{\alpha\beta}$, $\tilde{u}''_\alpha$, $\tilde{\tau}''_{\alpha\beta}$, u_α, $\tau_{\alpha\beta}$—together with the boundary conditions (1.5) and the continuity conditions (1.14). This system of equations evidently admits a simple interpretation in terms of an ordinary inclusion problem in the theory of generalized plane stress: it characterizes the thickness averages of displacement and stress appropriate to three elastic sheets that occupy the domains $\tilde{D}'$, $\tilde{D}''$, $\hat{D}$, respectively, are joined in perfect bond along the common edges C'_2, C''_2, and are (in the absence of body forces) subjected to the edge tractions t'_α and t''_α along C'_1 and C''_1, respectively; further the respective thickness, shear modulus, and Poisson's ratio of the three sheets referred to are given by (h', μ', ν'), (h'', μ'', ν''), and (h, μ, ν). In particular, the sheet occupying the original domain of adhesion $\hat{D}$ may be regarded as an elastic inclusion of fictitious thickness h and fictitious elastic constants μ, ν, whose periphery is bonded to the remaining two sheets; the thicknesses and elastic properties of the latter coincide with those of the two sheets S', S'' in the original load-transfer problem.

From (1.11), (1.12) follows readily

$$\left.\begin{aligned} \min[\mu', \mu''] \le \mu \le \max[\mu', \mu''],\\ \min[\nu', \nu''] \le \nu \le \max[\nu', \nu'']. \end{aligned}\right\} \tag{1.15}$$

Therefore, if—as we assume to be the case—the original elastic constants satisfy the inequalities

$$\mu' > 0, \qquad \mu'' > 0, \qquad -1 < \nu' < 1/2, \qquad -1 < \nu'' < 1/2, \tag{1.16}$$

one has

$$\mu > 0, \qquad -1 < \nu < 1/2. \tag{1.17}$$

Accordingly the strain-energy density associated with each of the three sheets involved in the foregoing inclusion problem is necessarily positive definite so that the solution of this auxiliary problem is unique except for an arbitrary additive infinitesimal rigid displacement of the entire composite body.†

The inclusion problem, to which we have been led, may be attacked by various standard methods of two-dimensional elastostatics. Once its solution has been found, the displacements and stresses in the unattached portions of S' and S'' in the original adhesion problem are known. The original unknowns appropriate to the bonded overlapping parts of S' and S'' are then directly computable. Indeed, (1.13), (1.3), in view of (1.9), (1.10), (1.11), (1.12), yield

† The uniqueness of the solution is of course contingent upon suitable regularity assumptions concerning the regions $\tilde{D}'$, $\tilde{D}''$, $\hat{D}$ and the associated elastostatic fields.

the sheet stresses

$$\hat{\tau}'_{\alpha\beta} = \frac{\mu'}{\mu}(\tau_{\alpha\beta}+\rho\eta\delta_{\alpha\beta}\tau_{\gamma\gamma}),\ \hat{\tau}''_{\alpha\beta} = \frac{\mu''}{\mu}(\tau_{\alpha\beta}-\eta\delta_{\alpha\beta}\tau_{\gamma\gamma}) \quad \text{on } \hat{D} \tag{1.18}$$

and the interior bond tractions

$$f_\alpha = \frac{\rho\eta h}{1+\rho}\tau_{\gamma\gamma,\alpha} \text{ on } \hat{D}, \tag{1.19}$$

provided

$$\eta = \frac{\nu'-\nu''}{(1+\nu')(1-\nu'')+(1+\nu'')(1-\nu')\rho}. \tag{1.20}$$

If, in particular, $\nu' = \nu''$, equations (1.19), (1.20) imply $f_\alpha = 0$ on $\hat{D}$ and we recover the result previously obtained by Goodier and Hsu [1]†: in this special instance the entire load transmission is effected by the edge bond-tractions defined in (1.8).

We now examine the manner in which the total force transmitted from the sheet S' to the sheet S'' is apportioned among the interior and the edge bond-forces acting on S''. Let R_α be the components of the resultant force appropriate to the *given edge loading* applied to S'. Then, because of (1.7),

$$R_\alpha = h'\int_{C'_1} t'_\alpha \, ds = -h''\int_{C''_1} t''_\alpha \, ds. \tag{1.21}$$

Next, denote by P_α and Q_α the components of the resultant *edge bond-forces* exerted *on* S'' over the arcs C''_2 and C'_2, respectively (Fig. 1), and let F_α stand for the components of the resultant *interior bond force* exerted *on* S''. Accordingly,

$$\left.\begin{aligned} P_\alpha = h''\int_{C''_2} \hat{\tau}''_{\alpha\beta}n_\beta \, ds, \qquad Q_\alpha &= -h'\int_{C'_2} \hat{\tau}'_{\alpha\beta}n_\beta \, ds, \\ F_\alpha = \int_{\hat{D}} f_\alpha \, dA. \end{aligned}\right\} \tag{1.22}$$

From (1.21), the inclusion analogy, and elementary equilibrium considerations, follows

$$R_\alpha = h\int_{C''_2} \tau_{\alpha\beta}n_\beta \, ds = -h\int_{C'_2} \tau_{\alpha\beta}n_\beta \, ds. \tag{1.23}$$

On the other hand, substitution from (1.18) into the first two of (1.22) and subsequent use of (1.23), (1.10), (1.11), (1.12), upon another appeal to (1.18), easily yield

$$\left.\begin{aligned} P_\alpha &= \frac{\rho}{1+\rho}R_\alpha - \frac{\rho\eta h'}{1+2\rho\eta}\int_{C''_2} \hat{\tau}'_{\gamma\gamma}n_\alpha \, ds, \\ Q_\alpha &= \frac{1}{1+\rho}R_\alpha - \frac{\rho\eta h'}{1+2\rho\eta}\int_{C'_2} \hat{\tau}'_{\gamma\gamma}n_\alpha \, ds. \end{aligned}\right\} \tag{1.24}$$

† Note that this conclusion follows also directly from (1.3) and (1.4).

Also, from the last of (1.22), (1.19), the divergence theorem, and (1.18), (1.10), (1.11), (1.12), one has

$$F_\alpha = \frac{\rho\eta h'}{1+2\rho\eta}\int_{\hat{C}} \hat{\tau}'_{\gamma\gamma} n_\alpha \, ds, \tag{1.25}$$

where $\hat{C}$ denotes the boundary of $\hat{D}$, i.e. $\hat{C} = C'_2 + C''_2$. We note that (1.24), (1.25) imply the relation

$$P_\alpha + Q_\alpha + F_\alpha = R_\alpha, \tag{1.26}$$

which also follows directly from the equilibrium of the sheet S'' and that of the composite assembly. If, in particular, $\nu' = \nu''$ one has $\eta = 0$ according to (1.20). In this instance (1.24), (1.25) reduce to

$$P_\alpha = \frac{\rho}{1+\rho} R_\alpha, \qquad Q_\alpha = \frac{1}{1+\rho} R_\alpha, \qquad F_\alpha = 0, \tag{1.27}$$

which are consistent with results obtained by Goodier and Hsu [1].†

We have assumed so far that both sheets, S' and S'', are of finite extent and that each of the boundaries C' and C'' is a simple closed curve; further, we have required the applied loading to consist of edge tractions exclusively. The entire preceding analysis—with obvious reinterpretations—is equally valid if D' and D'' are bounded but *multiply connected domains*; further, the generalization of the present analysis to loadings that include body forces is immediate. If either D' or D'' is an *unbounded region*, the boundary conditions (1.5) must be supplemented by suitable regularity conditions at infinity. While the inclusion analogy continues to hold in these circumstances, conditions (1.7) are here no longer necessarily satisfied‡ and the development starting with (1.21) may be in need of modification. Next, the extension of the inclusion analogy to *multiple bonded sheet assemblies*, involving possibly the mutual adhesion of several distinct sheets within common overlapping sheet domains, offers no difficulties whatsoever.

Finally, we note that the present treatment covers also the case in which one of the sheets considered, say S', is replaced by two separate sheets—each of the same material and middle section as S' but of half the original thickness—which are attached symmetrically to the two faces of S'' over the original domain of adhesion. Indeed, this mode of attachment renders the analysis more realistic since it tends to diminish bending effects, which have been left out of account.

2. APPLICATION: LOAD TRANSFER BETWEEN TWO SEMI-INFINITE BONDED SHEETS

In this section we apply the inclusion analogy developed in Section 1—suitably modified to accommodate unbounded sheet domains and body-force loads—to a particular problem concerning the load transfer between two overlapping semi-infinite sheets.

† See equation 17 in [1].

‡ In this instance the loading applied to $C'_1 + C''_1$ may be balanced by the resultant of the tractions at infinity even if all stresses vanish at infinity. See the example treated later in this paper.

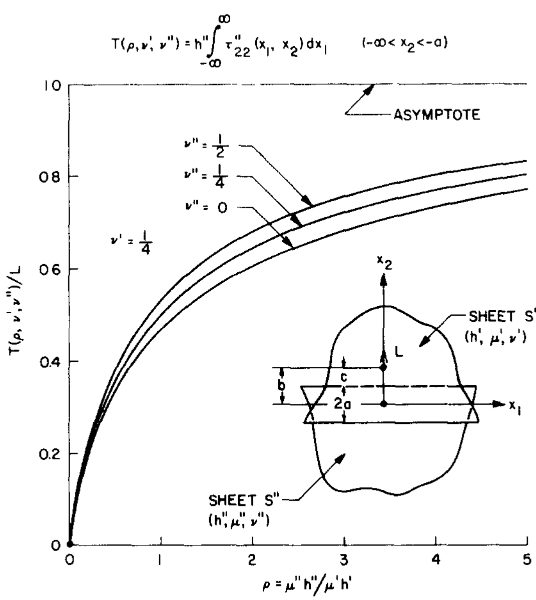

FIG. 2. Dependence of transmitted force on stiffness ratio.

Thus let the domains occupied by the sheets S' and S'' be defined by (see the inset diagram in Fig. 2)

$$\left.\begin{aligned} D' &= \{(x_1, x_2) | -\infty < x_1 < \infty, -a < x_2 < \infty\}, \\ D'' &= \{(x_1, x_2) | -\infty < x_1 < \infty, -\infty < x_2 < a\}. \end{aligned}\right\} \tag{2.1}$$

In this instance the domain of adhesion obeys

$$\hat{D} = \{(x_1, x_2) | -\infty < x_1 < \infty, -a < x_2 < a\}, \tag{2.2}$$

while the free sheet domains are characterized by

$$\left.\begin{aligned} \tilde{D}' &= \{(x_1, x_2) | -\infty < x_1 < \infty, a < x_2 < \infty\}, \\ \tilde{D}'' &= \{(x_1, x_2) | -\infty < x_1 < \infty, -\infty < x_2 < -a\}. \end{aligned}\right\} \tag{2.3}$$

Further, the boundaries of D' and D'' now become

$$\left.\begin{aligned} C' &= \{(x_1, x_2) | -\infty < x_1 < \infty, x_2 = -a\}, \\ C'' &= \{(x_1, x_2) | -\infty < x_1 < \infty, x_2 = a\}, \end{aligned}\right\} \tag{2.4}$$

with

$$C_2' = C', \qquad C_2'' = C'', \tag{2.5}$$

there being no unattached portions of the sheet boundaries in the present circumstances. We assume that the external loading applied to S' consists solely of a concentrated load of magnitude L with the point of application $(0, b)$ $(b > a)$, acting in the positive x_2-direction. On the other hand, S'' is to be entirely free of applied loads. As before, we seek the thickness averages of the displacements and stresses induced in S' and S''.

With a view toward removing in advance the singularity at the point of application of the load we set

$$\tilde{u}'_\alpha = \mathring{u}_\alpha + \overset{*}{u}'_\alpha, \qquad \tilde{\tau}'_{\alpha\beta} = \mathring{\tau}_{\alpha\beta} + \overset{*}{\tau}'_{\alpha\beta} \text{ on } \tilde{D}', \tag{2.6}$$

where $\mathring{u}_\alpha$, $\mathring{\tau}_{\alpha\beta}$ are the displacements and stresses of the classical generalized plane-stress solution† appropriate to a sheet of the same thickness and elastic properties as S' that occupies the *entire* plane and is subjected to the given concentrated load. Accordingly, $\mathring{u}_\alpha$ and $\mathring{\tau}_{\alpha\beta}$, at all points of $\tilde{D}'$ other than the singular point (0, b), are given by

$$\left.\begin{aligned}
\mathring{u}_1(x_1, x_2) &= \frac{(1+\nu')L}{8\pi\mu' h'}\frac{x_1(x_2-b)}{r^2}, \\
\mathring{u}_2(x_1, x_2) &= -\frac{L}{8\pi\mu' h'}\left[(3-\nu')\log r - (1+\nu')\frac{(x_2-b)^2}{r^2}\right], \\
\mathring{\tau}_{11}(x_1, x_2) &= \frac{L}{4\pi h'}\left[(1-\nu')\frac{x_2-b}{r^2} - 2(1+\nu')\frac{x_1^2(x_2-b)}{r^4}\right], \\
\mathring{\tau}_{22}(x_1, x_2) &= -\frac{L}{4\pi h'}\left[(3+\nu')\frac{x_2-b}{r^2} - 2(1+\nu')\frac{x_1^2(x_2-b)}{r^4}\right], \\
\mathring{\tau}_{12}(x_1, x_2) &= -\frac{L}{4\pi h'}\left[(1-\nu')\frac{x_1}{r^2} + 2(1+\nu')\frac{x_1(x_2-b)^2}{r^4}\right],
\end{aligned}\right\} \tag{2.7}$$

in which

$$r = \sqrt{[x_1^2 + (x_2-b)^2]}. \tag{2.8}$$

The load-transfer problem to be solved requires the determination of the unknowns $\overset{*}{u}'_\alpha$, $\overset{*}{\tau}'_{\alpha\beta}$, $\hat{u}'_\alpha$, $\hat{\tau}'_{\alpha\beta}$, $\tilde{u}''_\alpha$, $\tilde{\tau}''_{\alpha\beta}$, $\hat{u}''_\alpha$, $\hat{\tau}''_{\alpha\beta}$, f_α (all of which are to be regular throughout their domains of definition) such that (1.2), (1.3), (1.4) hold with $\tilde{u}'_\alpha$, $\tilde{\tau}'_{\alpha\beta}$ replaced by $\overset{*}{u}'_\alpha$, $\overset{*}{\tau}'_{\alpha\beta}$ and consistent with the continuity conditions (1.6). The boundary conditions (1.5) at present‡ give way to the regularity conditions at infinity

$$\overset{*}{\tau}'_{\alpha\beta} = o(1), \qquad \hat{\tau}'_{\alpha\beta} = o(1), \qquad \tilde{\tau}''_{\alpha\beta} = o(1), \qquad \hat{\tau}''_{\alpha\beta} = o(1) \qquad \text{as } x_\alpha x_\alpha \to \infty, \tag{2.9}$$

which express the vanishing of all stresses at infinity.

It is clear from the analysis in Section 1 that the foregoing adhesion problem is at once reducible to an associated inclusion problem in which the overlapping portions of S' and S'' are replaced by a single fictitious sheet with the properties (1.10), (1.11), occupying the strip $\hat{D}$. The edges C' and C'' of this inclusion are to be continuously attached to two sheets having the same thickness and elastic properties as S' and S'' but extending merely over the original free sheet domains $\tilde{D}'$ and $\tilde{D}''$, respectively. Further, the original loading is to be retained.

Adhering to the notation of Section 1 and with reference to the geometric agreements (2.1) to (2.5), as well as to the decomposition (2.6), (2.7), the auxiliary inclusion problem just described may be formulated as follows. We are to find the unknowns $\overset{*}{u}'_\alpha$, $\overset{*}{\tau}'_{\alpha\beta}$, u_α, $\tau_{\alpha\beta}$, $\tilde{u}''_{\alpha\beta}$, $\tilde{\tau}''_{\alpha\beta}$ such that (1.2), (1.13) hold with $\tilde{u}'_\alpha$, $\tilde{\tau}'_{\alpha\beta}$ replaced by $\overset{*}{u}'_\alpha$, $\overset{*}{\tau}'_{\alpha\beta}$, subject to the continuity

† See, for example, Girkmann [3], p. 115.

‡ See (2.5).

conditions (1.14), which now become

$$\left.\begin{aligned} \mathring{u}_\alpha(x_1, a) + \overset{*}{u}{}'_\alpha(x_1, a) &= u_\alpha(x_1, a) \qquad (-\infty < x_1 < \infty), \\ h'\mathring{\tau}_{\alpha 2}(x_1, a) + h'\overset{*}{\tau}{}'_{\alpha 2}(x_1, a) &= h\tau_{\alpha 2}(x_1, a) \qquad (-\infty < x_1 < \infty), \\ \tilde{u}''_\alpha(x_1, -a) &= u_\alpha(x_1, -a) \qquad (-\infty < x_1 < \infty), \\ h''\tilde{\tau}''_{\alpha 2}(x_1, -a) &= h\tau_{\alpha 2}(x_1, -a) \qquad (-\infty < x_1 < \infty), \end{aligned}\right\} \tag{2.10}$$

and the regularity conditions at infinity

$$\overset{*}{\tau}{}'_{\alpha\beta} = o(1), \qquad \tau_{\alpha\beta} = o(1), \qquad \tilde{\tau}''_{\alpha\beta} = o(1) \qquad \text{as } x_\alpha x_\alpha \to \infty. \tag{2.11}$$

The remaining original unknowns $\tilde{u}'_\alpha, \tilde{\tau}'_{\alpha\beta}, \hat{u}'_\alpha, \hat{\tau}'_{\alpha\beta}, \hat{u}''_\alpha, \hat{\tau}''_{\alpha\beta}, f_\alpha$ then follow directly from (2.6), (2.7) together with the first of (1.9) and (1.18), (1.19).

The foregoing auxiliary boundary-value problem is conveniently attached by means of Airy's stress function. Indeed, the complete solution of the governing stress equations of equilibrium and compatibility admits the representation

$$\left.\begin{aligned} \overset{*}{\tau}{}'_{\alpha\beta} &= \varepsilon_{\gamma\alpha}\varepsilon_{\rho\beta}\phi'_{,\gamma\rho} \text{ on } \tilde{D}', \\ \tau_{\alpha\beta} &= \varepsilon_{\gamma\alpha}\varepsilon_{\rho\beta}\phi_{,\gamma\rho} \text{ on } \hat{D}, \\ \tilde{\tau}''_{\alpha\beta} &= \varepsilon_{\gamma\alpha}\varepsilon_{\rho\beta}\phi''_{,\gamma\rho} \text{ on } \tilde{D}'', \end{aligned}\right\} \tag{2.12}†$$

$$\nabla^4\phi' = 0 \text{ on } \tilde{D}', \qquad \nabla^4\phi = 0 \text{ on } \hat{D}, \qquad \nabla^4\phi'' = 0 \text{ on } \tilde{D}'', \tag{2.13}$$

in which ∇^4 is the biharmonic operator. Since $\tilde{D}'$, $\hat{D}$, and $\tilde{D}''$ are simply connected regions, (2.12), (2.13), assure the integrability of the displacement–stress relations, which may now be written as

$$u_{(\alpha,\beta)} = \frac{1}{2\mu}\left[\varepsilon_{\gamma\alpha}\varepsilon_{\rho\beta}\phi_{,\gamma\rho} - \frac{\nu}{1+\nu}\delta_{\alpha\beta}\nabla^2\phi\right] \text{ on } \hat{D}, \text{ etc.} \tag{2.14}‡$$

Consequently the problem at hand reduces to the determination of solutions to (2.13) such that the stresses (2.12) and the displacements associated with these stresses through (2.14) conform to conditions (2.10), (2.11).

The preceding problem, in turn, is readily solved with the aid of the exponential Fourier transform. Adopting the notation

$$\mathbf{G}(x_2, s) = \int_{-\infty}^{\infty} G(x_1, x_2)\exp(isx_1)\,dx_1 \qquad (-\infty < s < \infty) \tag{2.15}$$

and using the transform to remove the x_1-dependence from (2.13), (2.12), (2.14), (2.10), (2.11), one arrives *at*§:

$$\left.\begin{aligned} \boldsymbol{\phi}'_{,2222} - 2s^2\boldsymbol{\phi}'_{,22} + s^4\boldsymbol{\phi}' &= 0 \qquad (a < x_2 < \infty, -\infty < s < \infty), \\ \boldsymbol{\phi}_{,2222} - 2s^2\boldsymbol{\phi}_{,22} + s^4\boldsymbol{\phi} &= 0 \qquad (-a < x_2 < a, -\infty < s < \infty), \\ \boldsymbol{\phi}''_{,2222} - 2s^2\boldsymbol{\phi}''_{,22} + s^4\boldsymbol{\phi}'' &= 0 \qquad (-\infty < x_2 < -a, -\infty < s < \infty), \end{aligned}\right\} \tag{2.16}$$

† Recall that $\varepsilon_{\alpha\beta}$ represents the components of the two-dimensional alternator.
‡ For the sake of brevity we suppress here the companion relations appropriate to the domains $\tilde{D}'$ and $\tilde{D}''$.
§ See Sneddon [4], Art. 45.

$$\left.\begin{aligned}\tau_{11} = \phi_{,22}, \qquad \tau_{22} = -s^2\phi, \qquad \tau_{12} = is\phi_{,2}\\ (-a < x_2 < a, \ -\infty < s < \infty), \text{ etc.}\end{aligned}\right\} \tag{2.17†}$$

$$\left.\begin{aligned}2\mu\mathbf{u}_1 &= \frac{i}{(1+\nu)s}[\nu s^2\phi+\phi_{,22}],\\ 2\mu\mathbf{u}_2 &= \frac{1}{(1+\nu)s^2}[\phi_{,222}-(2+\nu)s^2\phi_{,2}]\\ &(-a < x_2 < a, -\infty < s < \infty), \text{ etc.}\end{aligned}\right\} \tag{2.18}$$

$$\left.\begin{aligned}\mathring{\mathbf{u}}_\alpha(a,s)+\mathring{\tilde{\mathbf{u}}}'_\alpha(a,s) &= \mathbf{u}_\alpha(a,s) \qquad (-\infty < s < \infty),\\ h'\mathring{\boldsymbol{\tau}}_{\alpha 2}(a,s)+h'\mathring{\tilde{\boldsymbol{\tau}}}'_{\alpha 2}(a,s) &= h\boldsymbol{\tau}_{\alpha 2}(a,s) \qquad (-\infty < s < \infty),\\ \tilde{\mathbf{u}}''_\alpha(-a,s) &= \mathbf{u}_\alpha(-a,s) \qquad (-\infty < s < \infty),\\ h''\tilde{\boldsymbol{\tau}}''_{\alpha 2}(-a,s) &= h\boldsymbol{\tau}_{\alpha 2}(-a,s) \qquad (-\infty < s < \infty),\end{aligned}\right\} \tag{2.19}$$

$$\mathring{\tilde{\boldsymbol{\tau}}}'_{\alpha\beta} = o(1) \text{ as } x_2 \to \infty, \qquad \tilde{\boldsymbol{\tau}}''_{\alpha\beta} = o(1) \text{ as } x_2 \to -\infty \qquad (-\infty < s < \infty). \tag{2.20}$$

Further, from (2.7), (2.8) one finds ‡

$$\left.\begin{aligned}(-is)\mathring{\mathbf{u}}_1(a,s) &= -\frac{(1+\nu')Lc}{8\mu'h'}|s| \exp(-c|s|),\\ (-is)\mathring{\mathbf{u}}_2(a,s) &= -\frac{iL}{8\mu'h'}\left[(3-\nu')\frac{s}{|s|}+(1+\nu')cs\right]\exp(-c|s|),\\ \mathring{\boldsymbol{\tau}}_{22}(a,s) &= \frac{L}{4h'}[2+(1+\nu')c|s|]\exp(-c|s|),\\ \mathring{\boldsymbol{\tau}}_{21}(a,s) &= -\frac{iL}{4h'}\left[(1-\nu')\frac{s}{|s|}+(1+\nu')cs\right]\exp(-c|s|),\end{aligned}\right\} \tag{2.21}$$

provided

$$c = b-a. \tag{2.22}$$

The complete solution of the ordinary differential equations (2.16) that conforms to the regularity requirements (2.20) is given by

$$\left.\begin{aligned}\phi'(x_2,s) &= [A'_1(s)+(x_2-a)A'_2(s)]\exp(-|s|x_2),\\ \phi(x_2,s) &= [A_1(s)+x_2A_2(s)]\exp(-|s|x_2)\\ &\quad +[A_3(s)+x_2A_4(s)]\exp(|s|x_2),\\ \phi''(x_2,s) &= [A''_3(s)+(x_2+a)A''_4(s)]\exp(|s|x_2).\end{aligned}\right\} \tag{2.23}$$

† Here, as well as in (2.18), we omit the two companion relations holding for $(a < x_2 < \infty, -\infty < s < \infty)$ and $(-\infty < x_2 < -a, -\infty < s < \infty)$, respectively.

‡ See Erdélyi [5] p. 8 and p. 65.

The eight as yet arbitrary functions of s appearing in (2.23) need to be determined consistent with the eight transformed continuity conditions (2.19). This tedious but elementary computation, in which (2.17), (2.18), and (2.21) have to be used, leads to the results listed below.

$$\left.\begin{aligned}A_2(s) &= \frac{\mu L}{2\mu' h' \rho}\frac{\exp(-c|s|)}{|s|\Delta(as)}\left[2\beta_1\beta_2 a|s|\exp(a|s|)+\left(\frac{3-\nu'}{1+\nu'}+2c|s|\right)\exp(-a|s|)\varphi_2(as)\right],\\ A_4(s) &= \frac{\beta_2\mu L}{2\mu' h'}\frac{\exp(-c|s|)}{|s|\Delta(as)}\left[\exp(a|s|)\varphi_1(as)+2\left(\frac{3-\nu'}{1+\nu'}+2c|s|\right)a|s|\exp(-a|s|)\right],\end{aligned}\right\} \quad (2.24)$$

where

$$\left.\begin{aligned}\varphi_1(s) &= \frac{\beta_1}{\rho}\sinh(2|s|)+\frac{2}{1+\nu}\left[\beta_1\exp(2|s|)+\frac{1}{\rho}\exp(-2|s|)\right],\\ \varphi_2(s) &= \rho\beta_2\sinh(2|s|)+\frac{2}{1+\nu}[\beta_2\exp(2|s|)+\rho\exp(-2|s|)],\\ \Delta(s) &= \varphi_1(s)\varphi_2(s)-4\beta_1\beta_2 s^2,\\ \beta_1 &= \frac{4}{1+\nu'}+\frac{3-\nu'}{1+\nu'}\rho, \qquad \beta_2 = \frac{4}{1+\nu''}+\frac{3-\nu''}{1+\nu''}\frac{1}{\rho},\end{aligned}\right\} \quad (2.25)$$

while ρ is the stiffness ratio defined in (1.12). The remaining six desired functions of s involved in (2.23) are found to be expressible in terms of $A_2(s)$ and $A_4(s)$ as follows:

$$\left.\begin{aligned}A_1(s) &= \frac{1}{2|s|}\left\{(1+2a|s|)A_2(s)+\left[1-\frac{4}{(1+\nu)\beta_2}\right]\exp(-2a|s|)A_4(s)\right\},\\ A_3(s) &= -\frac{1}{2|s|}\left\{\left(1+\frac{4\rho}{1+\nu}\right)\exp(2a|s|)A_2(s)+(1-2a|s|)A_4(s)\right\},\\ A_1'(s) &= -\frac{L}{4h's^2}\left\{\frac{2[4+(3-\nu')\rho(\rho+2)]}{(1+\nu')\rho\beta_1}+(1+\nu')\frac{\rho}{\beta_1}c|s|\right\}\exp[-(c-a)|s|]\\ &\quad+\frac{2h}{(1+\nu)\rho\beta_1 h'|s|}[\rho A_2(s)+\beta_1\exp(2a|s|)A_4(s)],\\ A_2'(s) &= -\frac{(1+\nu')\rho L}{4\beta_1 h'|s|}\left[\frac{3-\nu'}{1+\nu'}+2c|s|\right]\exp[-(c-a)|s|]+\frac{4h}{(1+\nu)\beta_1 h'}A_2(s),\\ A_3''(s) &= -\frac{2\rho h}{(1+\nu)\beta_2 h''|s|}\left[\beta_2\exp(2a|s|)A_2(s)+\frac{1}{\rho}A_4(s)\right],\\ A_4''(s) &= \frac{4h}{(1+\nu)\beta_2 h''}A_4(s).\end{aligned}\right\} \quad (2.26)$$

Equations (2.23) to (2.26) fully determine the Fourier transforms $\boldsymbol{\phi}'$, $\boldsymbol{\phi}$, and $\boldsymbol{\phi}''$ of the three Airy stress functions introduced in (2.12). Consequently the corresponding displacement and stress transforms $\overset{*}{\mathbf{u}}{}'_\alpha$, $\overset{*}{\boldsymbol{\tau}}{}'_{\alpha\beta}$, $\mathbf{u}_\alpha$, $\boldsymbol{\tau}_{\alpha\beta}$, $\overset{*}{\mathbf{u}}{}''_\alpha$, $\overset{*}{\boldsymbol{\tau}}{}''_{\alpha\beta}$ are now fully known in view of (2.17) and (2.18). To obtain the *physical* antecedents of these transforms one invokes the inversion theorem, recalling that (2.15)—under suitable regularity assumptions—implies

$$G(x_1, x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{G}(x_2, s)\exp(-isx_1)\,ds \qquad (-\infty < x_1 < \infty). \tag{2.27}$$

In this manner one is, after trivial manipulations, led to real Fourier integral representations for $\overset{*}{\tilde{u}}'_\alpha$, $\overset{*}{\tilde{\tau}}'_{\alpha\beta}$, u_α, $\tau_{\alpha\beta}$, and $\tilde{u}''_\alpha$, $\tilde{\tau}''_{\alpha\beta}$. The solution of the auxiliary inclusion problem, given by $\tilde{u}'_\alpha$, $\tilde{\tau}'_{\alpha\beta}$, u_α, $\tau_{\alpha\beta}$, and $\tilde{u}''_\alpha$, $\tilde{\tau}''_{\alpha\beta}$, is now immediate from (2.6), (2.7). Finally, to complete the solution of the original load-transfer problem one appeals to (1.9) to (1.12) and (1.18) to (1.20).

The formal solution thus deduced is readily shown to conform to the given concentrated load and to satisfy the requisite field equations, bond conditions, continuity conditions, and regularity requirements at infinity. In the interest of brevity we refrain here from a complete explicit listing of the final results and from their aposteriori validation, which has been carried out in detail. Additional limiting checks on the results reached were supplied by the known solutions for a semi-infinite sheet under an internal concentrated load, the edge of the sheet being either free of tractions or rigidly held in place: the solution corresponding to a free edge is due to Melan [6]; that appropriate to a rigidly supported edge is inferred by specialization of results due to Frasier and Rongved [7].

We now cite from the solution found merely the explicit representations for the normal stresses acting parallel to the applied load in the free and bonded sheet portions and for the corresponding component of the interior bond tractions—all of which are of primary physical interest.

$$\left.\begin{aligned} \tilde{\tau}'_{22}(x_1, x_2) &= \mathring{\tau}_{22}(x_1, x_2) \\ &\quad - \frac{1}{\pi}\int_0^\infty [A'_1(s) + (x_2 - a)A'_2(s)]s^2 \exp(-sx_2)\cos(sx_1)\,ds \text{ on } \tilde{D}', \\ \tilde{\tau}''_{22}(x_1, x_2) &= -\frac{1}{\pi}\int_0^\infty [A''_3(s) + (x_2 + a)A''_4(s)]s^2 \exp(-sx_2)\cos(sx_1)\,ds \text{ on } \tilde{D}'', \end{aligned}\right\} \tag{2.28}$$

Next, (1.18) yields

$$\left.\begin{aligned} \hat{\tau}'_{22}(x_1, x_2) &= \frac{\mu'}{\mu}[\tau_{22}(x_1, x_2) + \rho\eta\tau_{\gamma\gamma}(x_1, x_2)] \text{ on } \hat{D}, \\ \hat{\tau}''_{22}(x_1, x_2) &= \frac{\mu''}{\mu}[\tau_{22}(x_1, x_2) - \eta\tau_{\gamma\gamma}(x_1, x_2)] \text{ on } \hat{D}, \end{aligned}\right\} \tag{2.29}$$

and τ_{22}, $\tau_{\gamma\gamma}$ are at present given by

$$\left.\begin{aligned} \tau_{22}(x_1, x_2) &= -\frac{1}{\pi}\int_0^\infty \{[A_1(s) + x_2A_2(s)]\exp(-sx_2) \\ &\quad + [A_3(s) + x_2A_4(s)]\exp(sx_2)\}s^2\cos(sx_1)\,ds \text{ on } \hat{D}, \\ \tau_{\gamma\gamma}(x_1, x_2) &= -\frac{2}{\pi}\int_0^\infty [A_2(s)\exp(-sx_2) - A_4(s)\exp(sx_2)]s\cos(sx_1)\,ds \text{ on } \hat{D} \end{aligned}\right\} \tag{2.30}$$

Finally, from (1.19) and the second of (2.30) one has

$$f_2(x_1, x_2) = \frac{2\rho\eta h}{(1+\rho)\pi}\int_0^{\infty} [A_2(s)\exp(-sx_2) + A_4(s)\exp(sx_2)]s^2\cos(sx_1)\,ds \text{ on } \hat{D}. \tag{2.31}$$

These results are rendered complete by the formula for $\mathring{\tau}_{22}$ in (2.7), by the previous results for A_1, A_2, A_3, A_4 and A'_1, A'_2, A''_3, A''_4 given in (2.24) to (2.26), and by the definitions of the parameters h, μ, ρ, η contained in (1.10), (1.11), (1.12), (1.20).

3. NUMERICAL RESULTS. DISCUSSION

We now discuss certain physically significant implications of the solution deduced in Section 2. In this connection our chief interest concerns the influence of the elastic properties upon the mechanism of the load transfer from the "upper" sheet S' to the "lower" sheet S'' and in particular upon the comparative role of the interior and the edge bond-tractions.

To this end we introduce first the following notation for the resultant forces on sections of S' and S'' at right angles to the applied concentrated load:

$$\left.\begin{aligned} T'(x_2) &= h'\int_{-\infty}^{\infty} \tau'_{22}(x_1, x_2)\,dx_1 \qquad (-a \le x_2 < a,\quad a < x_2 < b,\quad b < x_2 < \infty),\\ T''(x_2) &= h''\int_{-\infty}^{\infty} \tau''_{22}(x_1, x_2)\,dx_1 \qquad (-\infty < x_2 < -a,\quad -a < x_2 \le a). \end{aligned}\right\} \tag{3.1}†$$

Further, we set

$$T''(a) = P, \qquad T'(-a) = Q, \qquad F = \int_{-a}^{a}\int_{-\infty}^{\infty} f_2(x_1, x_2)\,dx_1\,dx_2, \tag{3.2}‡$$

so that P and Q stand for the resultant edge bond-forces exerted on S'' at $x_2 = a$ and $x_2 = -a$, respectively, whereas F is the resultant interior bond force§ acting on S''.

The evaluation of the integrals in (3.1), (3.2) is readily carried out on the basis of (1.1) and (2.28) to (2.31) with the aid of the identity‖

$$g(0) = \frac{1}{\pi}\int_0^{\infty} du\int_{-\infty}^{\infty} g(t)\cos(ut)\,dt, \tag{3.3}$$

which holds true for every function g that is continuously differentiable and absolutely

† Recall (1.1).

‡ Cf. the definitions (1.22).

§ Note that T', T'', P, Q, and F actually represent scalar force *components* parallel to the applied load; the corresponding components perpendicular to the load vanish by virtue of symmetry.

‖ See Titchmarsh [8], p. 13.

integrable on $(-\infty, \infty)$. This computation yields for T' and T'' the step functions given by

$$\left.\begin{aligned} T'(x_2; \rho, \nu', \nu'') &= \begin{cases} T(\rho, \nu', \nu'') - L & (b < x_2 < \infty) \\ T(\rho, \nu', \nu'') & (a < x_2 < b) \\ \dfrac{1}{1+\rho} T(\rho, \nu', \nu'') + K(\rho, \nu', \nu'') & (-a \le x_2 < a), \end{cases} \\ T''(x_2; \rho, \nu', \nu'') &= \begin{cases} \dfrac{\rho}{1+\rho} T(\rho, \nu', \nu'') - K(\rho, \nu', \nu'') & (-a < x_2 \le a) \\ T(\rho, \nu', \nu'') & (-\infty < x_2 < -a), \end{cases} \end{aligned}\right\} \quad (3.4)\dagger$$

where ρ is the stiffness ratio defined in (1.12) and

$$\left.\begin{aligned} T(\rho, \nu', \nu'') &= \frac{L}{2}\left[1 - \frac{1+\nu'}{1+\nu'+(3-\nu')\rho} + \frac{(1+\nu'')\rho}{3-\nu''+(1+\nu'')\rho}\right], \\ K(\rho, \nu', \nu'') &= \frac{(\nu'-\nu'')\rho L}{2(1+\rho)[1-\nu''+(1-\nu')\rho]}\left[1 - \frac{3-\nu'}{1+\nu'+(3-\nu')\rho} + \frac{(3-\nu'')\rho}{3-\nu''+(1+\nu'')\rho}\right]. \end{aligned}\right\} \quad (3.5)$$

From (3.4) and the first two of (3.2) one has for the resultant edge bond-forces

$$\left.\begin{aligned} P(\rho, \nu', \nu'') &= \frac{\rho}{1+\rho} T(\rho, \nu', \nu'') - K(\rho, \nu', \nu''), \\ Q(\rho, \nu', \nu'') &= \frac{1}{1+\rho} T(\rho, \nu', \nu'') + K(\rho, \nu', \nu''). \end{aligned}\right\} \quad (3.6)$$

As is apparent from the last of (3.4), $T(\rho, \nu', \nu'')$ represents the *total force transmitted* from the upper to the lower sheet, a portion of the load L being balanced by the stress resultant at infinity in S'. Since T is independent of a and b, the total transmitted force is not affected by the width of the overlapping strip of adhesion or by the location of the point of application of the load, as long as the latter lies within the free portion of S'. Consequently, the first of (3.5) gives the force transmitted also in case the concentrated load is replaced by a statically equivalent distributed loading applied to the unattached part of the upper sheet. It is clear from (3.5) that, for fixed ν' and ν'', $T(\rho, \nu', \nu'')$ is a steadily increasing function of the stiffness-ratio ρ. Further,

$$T(0, \nu', \nu'') = 0, \qquad \lim_{\rho \to \infty} T(\rho, \nu', \nu'') = L, \quad (3.7)$$

as is at once plausible in view of (1.12). Figure 2 depicts the total force transmitted as a function of the stiffness ratio for $\nu' = \frac{1}{4}$ and three values of the Poisson ratio ν'' appropriate to the lower sheet.

Turning to the discussion of the *resultant edge bond-forces* defined in (3.2), we observe from (3.6) that

$$P(\rho, \nu', \nu'') + Q(\rho, \nu', \nu'') = T(\rho, \nu', \nu''). \quad (3.8)$$

† For our present purpose it is helpful from here on to indicate explicitly the dependence upon the material parameters ρ, ν', ν'' of all functions considered.

Further, in view of the second of (3.5), $K(\rho, \nu', \nu')$ vanishes; hence (3.6), (1.12) furnish for the special case in which $\nu' = \nu''$,

$$\left.\begin{aligned} \frac{P(\rho, \nu', \nu')}{T(\rho, \nu', \nu')} &= \frac{\rho}{1+\rho} = \frac{\mu'' h''}{\mu' h' + \mu'' h''}, \\ \frac{Q(\rho, \nu', \nu')}{T(\rho, \nu', \nu')} &= \frac{1}{1+\rho} = \frac{\mu' h'}{\mu' h' + \mu'' h''}. \end{aligned}\right\} \quad (3.9)\dagger$$

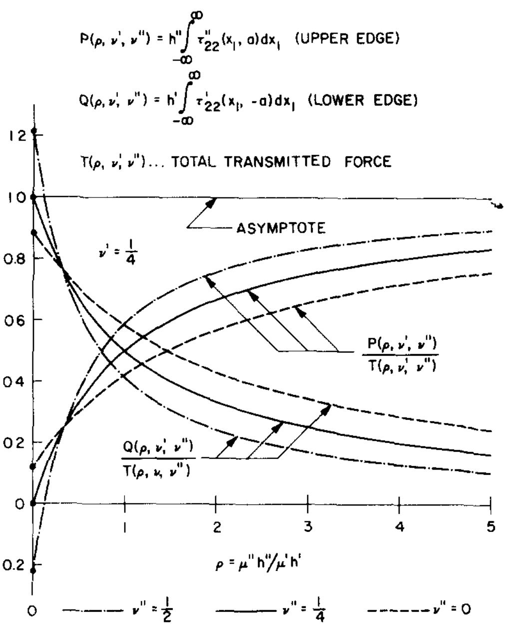

FIG. 3. Dependence of resultant edge bond-forces on stiffness ratio.

Figure 3 shows the resultant edge-bond forces P and Q, acting at the upper and lower edge respectively, in their dependence on the stiffness ratio ρ for $\nu' = \frac{1}{4}$ and several values of ν''. We emphasize that the functions represented here, like the curves in Fig. 2, are independent of the geometric parameters a and b.

Finally, we consider the *interior bond forces*, which—as shown in Section 1 and noted earlier by Goodier and Hsu [1]—vanish identically when both sheets have the same Poisson ratio. This is no longer the case if $\nu' \neq \nu''$, as is illustrated by Fig. 4, in which we plot $f_2(x_1, -a)$, $f_2(x_1, 0)$, and $f_2(x_1, a)$ for the material parameters $\rho = 1$, $\nu' = \frac{1}{4}$, $\nu'' = \frac{1}{2}$ and for $c = a$. It is interesting to observe that all three curves in Fig. 4 change sign. Indeed, the total area under each of these curves vanishes in accordance with

$$\int_{-\infty}^{\infty} f_2(x_1, x_2)\, dx_1 = 0 \qquad (-a \leq x_2 \leq a). \tag{3.10}$$

† Cf. the general formulas (1.27) for bounded sheets.

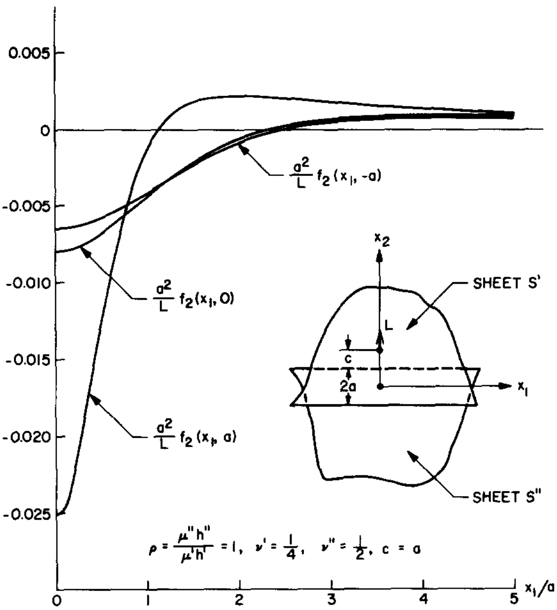

FIG. 4. Variation of interior bond-force component f_2 with x_1.

This relation may be inferred from (2.31) by means of (3.3); it also follows directly from elementary equilibrium considerations by virtue of the constancy of T' and T'' on the interval $(-a, a)$ implied by (3.4). From (3.10) follows further that the resultant interior bond force F, given by the last of (3.2), must vanish, as is also immediately apparent from (3.8). The foregoing self-equilibrance properties of the interior bond-force distribution represent a degeneracy peculiar to the particular problem at hand.

In order to gain some quantitative insight into the relative magnitude of the interior and the edge bond-forces, we compare next the x_2-component of the resultant interior bond force *per unit length of the strip of adhesion*,

$$\int_{-a}^{a} f_2(x_1, x_2)\,\mathrm{d}x_2 \qquad (-\infty < x_1 < \infty), \tag{3.11}$$

with the corresponding lineal intensity of the edge bond-forces given by

$$h''\tau''_{22}(x_1, a), \qquad h'\tau'_{22}(x_1, -a) \qquad (-\infty < x_1 < \infty). \tag{3.12}$$

Illustrative results pertaining to this comparison are presented in Figs. 5, 6, 7 for $c = a$ and various values of the governing material parameters ρ, ν', ν''. Figures 5 and 6 reveal the sensitive dependence on ν' and ν'' of the lineal *interior* bond-force density. In contrast, Fig. 7 indicates that this density is virtually independent of the stiffness ratio in the range of stiffness-ratios $\frac{1}{2} \leq \rho \leq 2$.

The solution deduced in Section 2 and discussed in Section 3 is easily extended—by finite superposition and an appropriate integration—to the case in which both sheets are subjected to a finite number of concentrated loads together with an arbitrary (sufficiently regular) body-force distribution, provided all loads are perpendicular to the sheet edges. In particular, the results we have established at once supply the results appropriate to a loading consisting of two equal and opposite collinear concentrated loads—one applied to

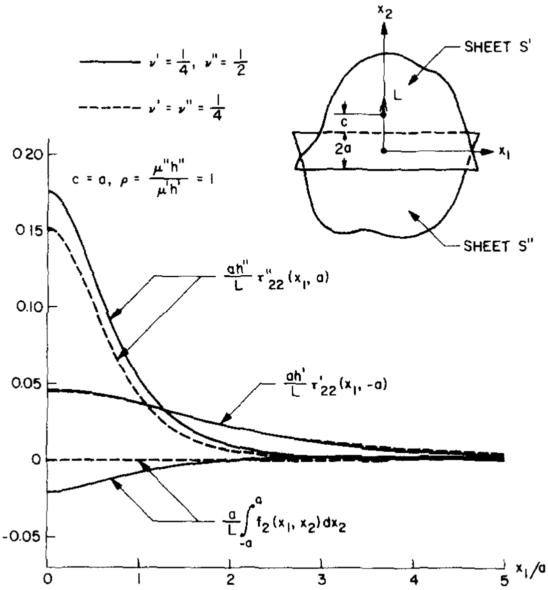

FIG. 5. Comparison of lineal intensity of interior and edge bond-forces.

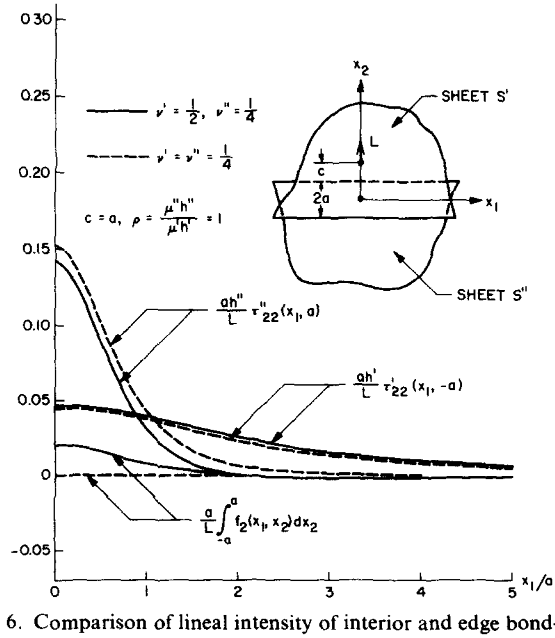

FIG. 6. Comparison of lineal intensity of interior and edge bond-forces.

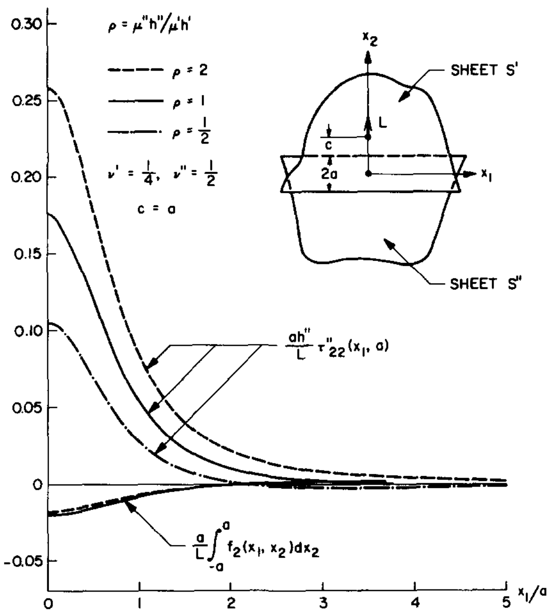

FIG. 7. Comparison of lineal intensity of interior and edge bond-forces.

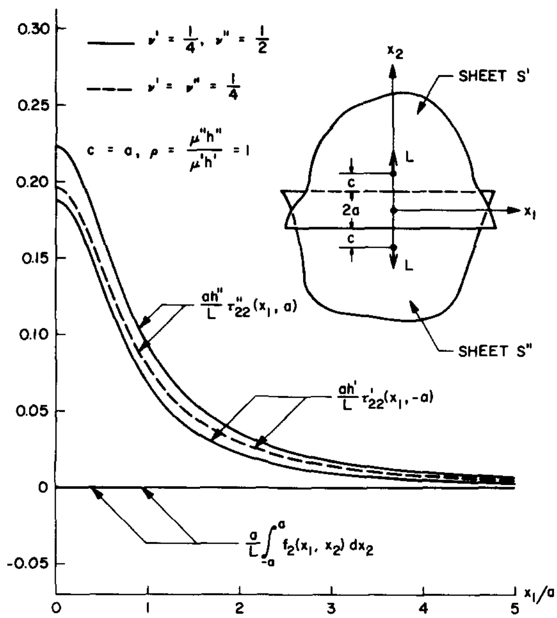

FIG. 8. Symmetric loading. Lineal intensity of interior and edge bond-forces.

each of the unattached sheet portions (see the inset diagram in Fig. 8). For such a symmetric loading the entire load acting on either sheet is transmitted to the other, no portion of the applied forces being resisted at infinity. The corresponding solution, in contrast to the basic singular solution discussed so far, refers to an idealization of a physically realistic test situation involving two sheets of sufficiently large but finite extent.

Figure 8, which is analogous to Figs. 5 and 6, depicts the influence of the Poisson-ratios upon the lineal intensity of the interior and the edge bond-force for the symmetric loading case. Although the interior bond force per unit width of the overlapping strip fails to vanish identically when $\nu' \neq \nu''$, its values are indistinguishable from zero on the scale of this drawing.

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Абстракт—Исследуется задача передачи нагрузки между двумя соединенными внахлестку, непрерывно связанными, упругими листами, разной толщины, обладающими различными свойствами материала, и нагруженными в своей плоскости. Указуется во первых, что указанный выже анализ напряженного состояния сводится, в пределах теории общенного плоского напряженного состояния, к задаче упругого включения. Используется эта общая сведенная схема к специфической задаче, касающейся передачи нагрузки между двумя соединенными внахлестку, полубесконечными листами, прикрепленными друг в другу вдоль простой полосы, примыкающей к их краям, когда один из листов находится под влиянием внутренной сосредоточенной силы, действующей под простыми углами к их краям. Исследуется детально полученное решение этой задачи, со специальным вниманием на качественную оценку сил на контуре, действующих повсюду внутри области сцепления, и связанных посредством краев двух рассматриваемых листов.